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DIMENSIONAL ANALYSIS AND
THE CONCEPT OF NATURAL UNITS
IN ENGINEERING

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ABSTRACT:

This monograph presents an introduction to dimensional analysis, a subject of profound significance for all quantitative sciences and engineering. Although the purely formal mathematical relations of dimensional analysis can be found in many texts, the present analysis, while still reasonably brief, goes well beyond the usual textbook treatment. It develops a unified theory of physical measurements which involves several novel and original features. Firstly, the usual qualitative concept of a physical dimension is replaced by a precise quantitative definition. Secondly, the famous Pi Theorem is shown to follow simply from the existance of a certain comprehensive set of consistent "natural units" which differ from ordinary English or metric units only in their fundamental scales of force, length, time, and temperature; these scales can always be chosen to fit the particular phenomena under consideration. Thirdly, it is shown that the mathematical form of any equation remains unaffected if all quantities in it be consistently transformed from ordinary fixed units to natural units. Fourthly, the analysis uncovers an unconventional dimensional relation (energy = mass x temperature) that is valid and useful under certain commonly encountered conditions. Finally, the theoretical discussion clears up certain common conceptual muddles, including those often associated with the various alternative units of force and mass.

Engineering applications of the theory are illustrated by various examples involving mainly fluid flow, turbomachinery, propellers, and related devices. In particular, highly significant absolute performance limits are established for certain broad classes of propellers, lifting rotors, and air turbines, merely by nondimensionalizing the well known momentum/energy relations in an unconventional but systematic manner.

Errata

for

Dimensional Analysis and the Concept
of Natural Units in Engineering by T. H. Gawain

p. 3 line 14: Change "Fl for work" to "FL for work".

p. 14 line 14: Change "F = FL" to "E = FL".

p. 49 line 7: Change "visosity" to "viscosity".

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1. Generalized Units

The various familiar quantities which appear in physical equations of all kinds are of two general types: dimensional or dimensionless. A dimensional quantity (for example, a force, a length, a velocity, a stress, a heat flux, etc.) is one which can only be expressed as some multiple of an appropriate physical unit of measurement. This numerical or algebraic multiple is termed the measure. The quantitative specification of any dimensional quantity, therefore, always involved both unit and measure. Moreover, for any dimensional quantity of fixed physical magnitude, the numerical value of the measure is inversely proportional to the size of the unit of measurement. Hence, the numerical measure of any dimensional quantity can never be dissociated from the magnitude of its corresponding unit.

On the other hand, a dimensionless quantity is simply one which is not associated with any particular physical unit. It is a pure number. Consequently, the numerical magnitude of any dimensionless quantity does not depend in any way on the units used for expressing other related dimensional quantities occurring in the same equation or problem. In particular, the ratio of any two dimensional quantities which are both expressed as multiples of a common unit turns out to be independent of the size of that unit. It is, therefore, a dimensionless number. For example, let symbols a and b denote two physical lengths, say the length and width of a certain wooden block. These are both dimensional because their respective numerical measures depend on the size of the length unit used. This might be any convenient unit of length: inches, feet, meters, etc. On the other hand, the ratio a/b turns out to have a

fixed value independent of the size of the length unit in which a and b are separately expressed. Hence, ratio a/b is dimensionless.

It often happens also that a quantity initially looks as if it might be dimensional but it turns out, upon further analysis, that its numerical magnitude is actually independent of the particular units which appear to be involved. This assumes, of course, that the actual units used are themselves consistent, as will be explained more fully later; for the present, we take the consistency of the units for granted. Under these circumstances we conclude that the quantity in question is actually dimensionless. This important idea is best clarified by means of a specific example. Suppose, for instance, that a certain situation involves a velocity V, a length ℓ, and the acceleration of gravity g. These are all dimensional quantities whose magnitudes depend on the units in which they happen to be expressed. Suppose further that quantities of the form $V\ell g$ and $V/\sqrt{\ell g}$ happen to appear in the mathematical analysis of the problem. Offhand, it might perhaps appear that the numerical values of both these combinations is dependent on the particular units in which the separate quantities V, ℓ, and g happen to be expressed. Analysis shows that the magnitude of $V\ell g$ does indeed depend on the units used. On the other hand, the numerical value of $V/\sqrt{\ell g}$ turns out to be independent of the units used, provided, of course, that they are consistent. Thus, for example, $V/\sqrt{\ell g}$ has the same numerical value whether evaluated in terms of English units or metric units. We conclude, therefore, that the quantity $V\ell g$ is dimensional but that the quantity $V/\sqrt{\ell g}$ is dimensionless.

We can now generalize the above important principle as follows: if it can be shown that the magnitude of some given quantity is in fact

independent of the actual magnitudes of particular units (provided only that all units are consistent), then the quantity in question is necessarily dimensionless. The idea of consistency of units as mentioned above is elaborated further in the later discussion.

For most engineering purposes, it suffices simply to label all dimensional quantities with the customary names or abbreviations of their respective units such as ft, sec, lb/ft^2, Btu/hr ft^2, and so forth. For purposes of dimensional analysis, however, it is considerably more useful to represent every such unit by a corresponding generalized symbol. Thus, the chosen unit of force may be represented by the letter F, the unit of mass by M, the unit of length by L, and so on. We term symbols such as F, M, L . . . generalized units or dimensions. Of course, we can also have composite generalized units such as L/T for velocity, F/L^2 for pressure, Fl for work, and so forth.

It should be pointed out that this particular usage of these dimensional symbols as generalized units is somewhat unorthodox. Most textbooks use these same symbols, which they also call dimensions, in only a qualitative (or even in only a metaphorical) sense. The precise quantitative significance associated with the present concept of the generalized unit turns out to be much clearer as well as far more useful than the essentially qualitative character of the orthodox concept of a dimension.

Two related advantages of this scheme are its generality and its exactness. Thus, written relations involving such generalized units remain valid and quantitatively exact whether the symbols subsequently are taken to represent English units, metric units, or even units which are neither English or metric! Of particular value and interest in this

connection are units of the type which we have chosen to call "natural units." These are explained in detail later in this report.

The present discussion encompasses the entire realm of mechanics and thermodynamics, but it specifically excludes the phenomena of electricity and magnetism. It is a remarkable fact that all of the generalized units, however complex, which can occur in this realm are expressible as various combinations of just six generalized units! These six fundamental dimensions are summarized in Table 1.1.

Table 1.1 Fundamental Dimensions

Quantity	Symbol for One Generalized Unit
Force	F
Mass	M
Length	L
Time	T
Energy (Including Heat)	E
Temperature	Θ

In principle, the phenomena of electricity and magnetism could also be included in this analysis by adding one more fundamental dimension to represent electrical charge, but this possibility will not be developed further in the present discussion.

Once the above six generalized units have been specified, the corresponding consistent derived units for all other quantities can be expressed in terms of these fundamental ones. Thus, for example, the consistent derived unit for velocity may be defined as the velocity which corresponds to unit length L traversed in unit time T. This may, therefore, be denoted by the generalized symbol L/T. For the particular

case where L happens to be one foot and T happens to be one second, the corresponding consistent derived unit of velocity then becomes one foot per second abbreviated as ft/sec. On the other hand, if L is one kilometer and T is one hour, then L/T denotes km/hr. Similarly, the consistent derived unit for pressure is that pressure which corresponds to unit force F acting over unit area L^2, or F/L^2. Again, this expression could represent lbf/ft^2, $dynes/cm^2$, $newtons/m^2$, and so on, depending on the specific fixed units adopted in a particular case. This method of expressing, in a generalized way, various typical derived units in terms of the six fundamental units is illustrated further in Table 1.2.

This initial formulation also suggests that the magnitudes of each of the six fundamental units may be specified arbitrarily. Consequently, the corresponding magnitudes of all consistent derived units become fixed accordingly. For convenience, we shall designate any hypothetical dimensional system which actually possesses six such degrees of freedom as an unconstrained system.

We shall see later, however, that the two general types of systems of major importance in science and engineering each satisfy two additional constraints so that in each case we end up with just four degrees of freedom. Naturally, these constraints ultimately affect the final form of the various units. These units are shown in Table 1.2 in their initial unconstrained form. The basis for the subsequent changes in the form of the units is developed in the following sections of this report.

Table 1.2 Typical Consistent Derived

Units in Unconstrained Form

		Typical Symbol	Generalized Unit	
1.	Angle	α	1	See Note
2.	Strain	ϵ	1	
3.	Area	A	L^2	
4.	Volume	v	L^2	
5.	Velocity	V	L/T	
6.	Angular Velocity	ω	1/T	
7.	Acceleration	a	L/T^2	
8.	Volumetric Flow Rate	Q	L^3/T	
9.	Pressure	p	F/L^2	
10.	Moment	M	FL	
11.	Surface Tension	σ	F/L	
12.	Viscosity	μ	FT/L^2	
13.	Kinematic Viscosity	υ	FLT/M	
14.	Specific Impulse	I	FT/M	
15.	Mass Flow Rate	$\dot{m}$	M/T	
16.	Mass Flux	G	M/TL^2	
17.	Moment of Inertia	J	ML^2	
18.	Density	ρ	M/L^3	
19.	Gas Constant	R	$FL/M\Theta$	
20.	Specific Weight	γ	F/L^3	
21.	Power	P	E/T	
22.	Energy Flux	q/A	E/TL^2	

23.	Specific Enthalpy	h	E/M
24.	Specific Heat	C_p	$E/M\theta$
25.	Specific Entropy	s	$E/M\theta$
26.	Thermal Conductivity	k	$E/TL\theta$
27.	Heat Transfer Coefficient	U	$E/TL^2\theta$
28.	Coefficient of Thermal Expansion	β	$1/\theta$
29.	Inertial Constant	k_I	ML/FT^2
30.	Work/Energy Factor	k_E	FL/E

NOTE: The symbol 1 signifies that the corresponding quantity is dimensionless.

2. Auxiliary Constraints

In the discussion so far, the six fundamental generalized units have been treated as independent. This amounts to saying that the magnitude of each one of these six may be prescribed arbitrarily, without any reference to the magnitudes assigned to any of the others. However, in the various systems of units which have actually won general acceptance in science and engineering, certain additional constraints are customarily imposed, thus reducing the number of arbitrary selections which can be made to less than six. We shall now summarize the most important of these constraints.

In some systems, the units F, M, L, and T are so chosen that unit force F imparts unit acceleration L/T^2 to unit mass M. We shall term any system of units which conforms to this constraint an inertial system.

Other systems are subject to a somewhat different constraint. In particular, the units F and M may be so chosen that unit force F equals the weight of unit mass M in the earth's gravitational field, under prescribed standard conditions of gravitational acceleration. We shall term any system of units which conforms to this constraint a gravitational system.

All systems of units to be considered further in this discussion are either inertial or gravitational systems, these two categories being quite distinct.

Another important constraint on dimensional systems is based on the character of the energy unit E. In the present context, energy may be expressed in terms either of work or of heat, that is, either in mechanical or in thermal units. The mechanical unit of energy E is defined as the work done by unit force F on unit displacement L. On the other hand, the thermal unit of energy E is defined as the heat required to produce unit

increase of temperature θ in unit mass M of water at prescribed standard conditions of pressure and temperature. The specification of water as the particular medium in connection with this last definition is arbitrary but convenient, and it is the universally accepted standard. In line with these definitions, we can described a system of units as being either <u>mechanical</u> or <u>thermal</u>, according to the units used for expressing energy. In particular, in a mechanical system <u>all</u> energy quantities must be expressed exclusively in mechanical units. On the other hand, if a system utilizes either thermal units exclusively, or else uses some mixture of mechanical and thermal units depending on the particular energy quantity in question, we shall classify it as a <u>thermal</u> system.

All systems of units to be considered further in this discussion are either mechanical or thermal systems, these two categories being here treated as distinct and mutually exclusive.

In mechanical and aeronautical engineering, for all general purposes other than those associated with the phenomena of electricity and magnetism, two types of dimensional systems have established themselves as preeminent. We shall call them <u>dynamic</u> and <u>thermodynamic</u> systems, respectively. These two distinct types of dimensional systems can now be simply defined as follows:

Any system of units which is both <u>inertial</u> and <u>mechanical</u> shall be here designated as a <u>dynamic</u> system.

Any system of units which is both <u>gravitational</u> and <u>thermal</u> shall be designated as a <u>thermodynamic</u> system.

The relationships defined above are summarized in Table 2.1. All further discussion herein shall be restricted to just these two main types of dimensional systems. It will be shown that for each of these two major types the fundamental units whose magnitude can be specified arbitrarily

are just four in number. This reduction from six degrees of freedom to four comes about because each of these types of systems is subject to two auxiliary constraints as explained above.

Both types of systems discussed above may, of course, be implemented in English units and in metric units. This is illustrated further in Table 2.2. The English units shown are those commonly used for engineering purposes in the English speaking countries of the world. The metric units shown are known as MKS units (meter, kilogram, second). They are used in engineering work throughout Europe. Also in common use for scientific purposes in all countries, are the metric CGS units (centimeter, gram, second), but these are perhaps of less importance in engineering; they are not included in the table.

Table 2.1 The Two Main Types

of Dimensional Systems

I. Dynamic System

1. Inertial Constraint

Unit force F imparts unit acceleration L/T^2 to unit mass M.

2. Mechanical Constraint

The unit of energy E equals the work done by unit force F over unit displacement L.

II. Thermodynamic System

1. Gravitational Constraint

Unit force F equals the weight of unit mass M under standard conditions of gravitational acceleration g_s L/T^2.

2. Thermal Constraint

The unit of energy E equals the heat required to produce unit temperature rise Θ in unit mass M of water at standard conditions of pressure and temperature.

Table 2.2 Fundamental Units in Several Common Systems

Quantity	Generalized Units	English Units		Metric Units	
	Symbol for One Unit	Dynamic	Thermodynamic	Dynamic	Thermodynamic
Force	F	lb	lb or lbf	newton*	kg or kgf*
Mass	M	slug*	lb or lbm*	kg	kg or kgm
Length	L	ft	ft	m	m
Time	T	sec	sec	sec	sec
Energy	E	ft lb*	Btu*	joule*	kcal*
Temperature	Θ	^{o}F	^{o}F	^{o}C	^{o}C
<u>Conversion Factors</u> Inertial Constant k_I	$(\frac{M}{F})\frac{L}{T^2}$	1	$32.1739(\frac{lbm}{lbf})\frac{ft}{sec^2}$	1	$9.80665(\frac{kgm}{kgf})\frac{m}{sec^2}$
Work/Energy Ratio k_E	$\frac{FL}{E}$	1	778.3 ft lbf/Btu	1	4186 joule/kcal

Note: Only four units in each column can be regarded as independently given. The two units marked with an asterisk in each column are regarded as consistent derived quantities.

3. Newton's Second Law of Motion

For any unconstrained system in which all four of the units F, M, L, and T are arbitrary and independent, Newton's Second Law of Motion may be written in the form:

$$f = \frac{1}{k_I} m\ a \qquad (3\text{-}1)$$

where:

f = net force acting on a body

m = mass of body

a = acceleration of body

k_I = inertial constant

The numerical magnitude of the inertial constant k_I in this equation depends solely on the relative magnitudes assigned to the four units F, M, L, and T. Also, from Equation (3-1) we may infer the units of k_I. Denoting these units by the symbol $U(k_I)$, we may write:

$$U(k_I) = \frac{ML}{FT^2} \qquad (3\text{-}2)$$

Next, consider any inertial system. From the definition of inertial units, it follows that units F, M, L, and T are so related in magnitude that:

$$k_I = 1 \qquad (3\text{-}3)$$

Moreover, in this case, the magnitude of k_I remains unity by definition regardless of how the magnitudes of M, L, and T or F, L, and T are assigned. Hence, by our original definition of a dimensionless quantity, it follows that in any inertial system, the inertial constant k_I is dimensionless.

Equation (3-1) now becomes simply:

$$f = m\ a \qquad (3\text{-}4)$$

Next, consider the case where f in this equation equals unit force F, and where m equals unit mass M. The resulting aceleration a then equals

unit acceleration $\frac{L}{T^2}$ and we may write

$$F = \frac{ML}{T^2} \tag{3-5}$$

also conversely

$$M = \frac{FT^2}{L} \tag{3-6}$$

These results express in quantitative form the fact that only three of the four units F, M, L, and T can now be regarded as independent. Thus, if we choose M, L, and T as the three independent units, then force F becomes the consistent derived unit as defined by Equation (3-5). Conversely, if we choose F, L, and T as the three independent units, then mass M becomes the consistent derived unit as defined by Equation (3-6).

From Equation (3-5), we also see that in any inertial system of units the generalized unit F can always be replaced by $\frac{ML}{T^2}$ and thus eliminated. Alternatively, from Equation (3-6), we see that M can always be replaced by $\frac{FT^2}{L}$ and thereby eliminated. By means of these substitutions, either the unit F or the unit M (but not both) can be eliminated from all of the original generalized units shown in Table 2.2. The resulting generalized units so obtained are summarized later in Table 5.2. Please note, however, that the important relations (3-5) and (3-6) apply to inertial systems only!

Next, consider the alternative to inertial units, namely, gravitational units. As we have already seen, in general the inertial constant k_I has the units $\frac{ML}{FT^2}$. In the gravitational case it turns out, however, that k_I becomes numerically equal to the standard acceleration of gravity g_s. Of course, g_s itself has units of acceleration $\frac{L}{T^2}$. To emphasize the numerical equality between k_I and g_s, but at the same time to mark the apparent disparity between the units of these two quantities, the inertial

constant k_I for the case of gravitational units is usually assigned the distinctive symbol g_o or g_c. In this text we use g_o.

Hence, for any gravitational system of units, Newton's Second Law of Motion assumes the specific form:

$$f = \frac{1}{g_o} \; m \; a \qquad (3\text{-}7)$$

In English and metric MKS Units, respectively, the inertial constant takes on the following values, namely:

$$g_o = 32.1739 \left(\frac{lbm}{lbf}\right)\left(\frac{ft}{sec^2}\right)$$

$$= 9.80665 \left(\frac{kgm}{kgf}\right)\left(\frac{m}{sec^2}\right) \qquad (3\text{-}8)$$

Note that the actual numerical magnitude of g_o depends only on the units of length and time involved. The ratio $\left(\frac{lbm}{lbf}\right)$ or $\left(\frac{kgm}{kgf}\right)$, in other words the ratio $\left(\frac{M}{F}\right)$, is included only for the sake of formal consistency of units in the equation of motion. We shall now show, however, that by a slight change of convention, this mass/force ratio can actually be deleted without introducing any error.

In the case of a body falling freely in a gravitational field of intensity g, the acceleration a becomes equal to g and the force f acting on the body becomes equal to its weight w. Hence, Equation (3-7) reduces to:

$$w = \frac{1}{g_o} \; m \; g \qquad (3\text{-}9)$$

Now, if the gravitational acceleration g be taken as equal to the standard acceleration g_s, then the corresponding weight w becomes what we may term the standard weight w_s. In this case, Equation (3-9) can be rewritten and rearranged to read:

$$\left(\frac{w_s}{m}\right) = 1 = \left(\frac{g_s}{g_o}\right) \qquad (3\text{-}10)$$

The most striking fact here revealed is that this ratio is numerically equal to unity for any gravitational system regardless of how the magnitudes of M, L, T, or of F, L, T happen to be chosen. This follows from the fact that standard weight w_s is always numerically equal to mass m by definition, and likewise that g_o is always numerically equal to g_s. Moreover, the numerical value of g_s itself depends only on the magnitudes of units L and T and is, of course, independent of units F or M.

Inasmuch as the numerical magnitude of the above ratio is unity and is independent of all units provided only that they constitute a gravitational system, then by the criterion we have earlier laid down, this ratio must be deemed to represent a dimensionless quantity in such a system. Hence, with the equation on the left side of (3-10) we can associate the dimensional statement:

$$\frac{F}{M} = 1 = \text{dimensionless!} \qquad (3\text{-}11)$$

Moreover, as a consequence of (3-11), the generalized units of g_o reduce as follows.

$$U(g_o) = \left(\frac{M}{F}\right)\left(\frac{L}{T^2}\right) = (1)\left(\frac{L}{T^2}\right) = \frac{L}{T^2} \qquad (3\text{-}12)$$

Furthermore, if we adhere consistently to this particular convention, it follows that the quantities f and m in Equation (3-7) must now be expressed in like units, the earlier distinction between unit F and unit M having disappeared by virtue of statement (3-11)!

It should now be clear that we have the choice of two formally distinct yet quantitatively equivalent conventions for any gravitational system, depending on whether we do or do not adopt the formalism represented by Equation (3-11). If we do choose to incorporate (3-11) within our system, and this is certainly a valid option, then there is no longer any need for distinguishing between the symbol F for unit force and the symbol M for

unit mass. A single common symbol will now serve for either use. It is well to choose for this purpose a new symbol which is not identified exclusively with either force or mass. For definiteness in the present discussion, we choose the symbol G to represent this generalized gravitational unit.

The two alternatives discussed above are summarized in Table 3.1.

Table 3.1

Alternative Conventions for Gravitational Units of Force and Mass

	Distinct Units		Common Unit
	Force	Mass	Force or Mass
Generalized Units	F	M	G
English Units	lbf	lbm	lb
Metric MKS Units	kgf	kgm	kg

Both of the above conventions are correct and both find wide use. Most textbooks favor the use of distinct units of force and mass since this preserves a certain parallel with the usage in an inertial system of units. On the other hand, many practicing scientists and engineers prefer the use of the common unit on the grounds of convenience and simplicity.

The particular convention finally adopted makes a nominal difference in the apparent units of certain quantities. For example, work per unit mass becomes ft lbf/lbm in the one notation and simply ft in the other.

Likewise, specific impulse becomes lbf sec/lbm in the one notation and simply sec in the other. The distinction is purely formal, however. The actual magnitudes of the respective units are entirely unaffected by the difference of notation. Corresponding numerical operations in the two conventions are always identical.

Some theoreticians are averse to the use of a common label for force and mass units on the grounds that force and mass are entirely dissimilar physically. This physical distinction is indeed fundamental. Nevertheless, the expression of unlike physical quantities in terms of units which are nominally alike, while somewhat exceptional, is not wholly without precedent. Perhaps the best examples of this are heat and work. These two quantities are so unlike in their physical manifestations that it required many years of intensive scientific effort to establish the fact that they are simply two different forms of energy in transit and that they can indeed be expressed in like units if desired. Also, work and moment are quite different physically, yet both are expressible in terms of the common generalized unit FL. Also, specific heat and specific entropy are physically distinct yet share the common unit $E/M\Theta$. Hence, the adoption of a common label for the gravitational units of force and mass does not really entail any denial of the important qualitative and physical difference between these two quantities. This usage arises instead from the fundamental fact that in any gravitational system, a definite choice of the magnitude of <u>either</u> unit suffices to determine the magnitudes of <u>both</u>. These two units are, therefore, not independent. It is in some respects simpler, clearer, and more consistent to avoid two distinct symbols in our list of fundamental units when in fact only one of these can be regarded as truly independent.

Now reverting again to the discussion of inertial systems, we invite attention once more to the familiar and important relation (3-5) which for convenience we repeat here, namely:

$$F = \frac{ML}{T^2} \qquad (3\text{-}5)$$

Under the conventional qualitative interpretation of dimensional symbols, this is generally taken to mean that symbol F, for example, somehow represents the "physical character or essence" of "mass times acceleration." Moreover, this "character" or "essence" denoted by F is conceived of as being an innate physical quality, fixed and independent of any particular unit or system of units in terms of which it may happen to be expressed in any particular instance. Of course, a similar interpretation is applied to the dimensional symbol for mass M, for acceleration L/T^2, and so on, each of which is supposed to retain its own unique and fixed qualitative character under all circumstances. Thus, the meaning of each symbol is taken to be immutable and quite independent, for example, of whether we happen to be dealing with an inertial or a gravitational system. In other words, dimensional symbols are regarded as somehow representing the ultimate being of Nature herself who remains indifferent to the arbitrary and contingent units adopted by Man.

Actually, whether we realize it or not, such an interpretation of dimensional symbols as representing unchanging physical qualities or essences is actually metaphysical; it has no operationally verifiable meaning. Furthermore, the difficulties that can arise from any attempt to work consistently within this conceptional framework are very great. For example, the scientist who has once accepted the inertial dimensional relation (3-5) in such a grandiose sense is then debarred from ever accepting the gravitational relation (3-11), <u>even for a gravitational</u>

system! We rewrite (3-11) here for convenience in the form:

$$F = M \qquad (3\text{-}11)$$

If confronted with this relation, he is inclined to dismiss it out of hand, as absurd. He feels that, to paraphrase Kipling, "Force is Force and Mass is Mass, and never the twain shall meet!"

Such paradoxes and confusions can be eliminated by dropping this metaphysical interpretation of dimensional symbols and adopting instead the simple concept of the generalized unit as proposed in this paper. This definition treats the dimensional symbol much like any other ordinary algebraic symbol. The symbol merely denotes precisely one unit of a particular kind. Dimensional relationships then take on exact quantitative meanings. Also, the specific forms of these relationships then become definitely contingent upon the particular type of dimensional system we happen to be dealing with. Therefore, certain dimensional relations like (3-5), for example, which happen to characterize any inertial system, need not necessarily agree with corresponding relations like (3-11) for a gravitational system. The seeming paradox is, therefore, resolved. Furthermore, as long as we adhere consistently to a system of a fixed type, say a dynamic system for example, then all dimensional relations remain invariant and independent of the specific magnitudes which may happen to be assigned to the independent fundamental units of the system.

4. Energy Relations

Recalling that the mechanical unit of energy E is simply the work done by unit force F acting over unit displacement L, we can at once write the following dimensional relation for any mechanical system, namely:

$$E = FL \qquad (4\text{-}1)$$

With regard to thermal systems, the situation is slightly more complicated. For heat addition to some arbitrary medium at constant pressure we may write, provided that the temperature rise is small,

$$q = C_p m \,\Delta T$$

q = net heat added

m = mass of medium

ΔT = temperature rise

C_p = specific heat of medium at constant pressure (4-2)

Now, consider the specific case where the medium happens to be water at the specified standard conditions of pressure and temperature. Also in this case, let m be taken as equal to unit mass M and let ΔT be taken as equal to unit temperature rise θ. Recall that the thermal unit of energy E is the heat required to produce unit temperature rise θ in unit mass M of water at standard conditions. Hence, in this case, q will simply equal unit energy E. In addition, from Equation (4-2) we note that for water at standard conditions, the specific heat C_p in any thermal system equals unity by definition, regardless of the actual magnitudes assigned to units M and θ. Consequently, specific heat C_p must be regarded as a dimensionless quantity in any thermal system! Upon substituting these various quantities into Equation (4-2) we finally obtain the basic dimensional relation

$$E = M\theta \qquad (4\text{-}3)$$

which is applicable to any thermal system of units.

Thus, the dimensional relation which governs the derived energy unit E is Equation (4-1) for any mechanical system or Equation (4-3) for any thermal system. In either case, E can be eliminated from the list of fundamental units. Hence, whenever E occurs in the generalized units listed in Table 1.2, it can be replaced by FL or Mθ, respectively, as appropriate.

It should be pointed out that whereas the mechanical relation (4-1) is universally recognized in textbook discussions of dimensional analysis, the corresponding thermal relation (4-3), while equally correct, is more or less unconventional and is not usually mentioned. In fact, the only textbook writer who has actually proposed the thermal relation (4-3), so far as this author is aware, is Csanady. See the bibliography, item (18). Even in that reference, however, there seems to be no clear statement of the fact that the stated relation is specifically applicable only to thermal systems of units, but not otherwise.

The distinction between a mechanical and a thermal system can also be conveniently expressed in terms of a work/energy conversion factor k_E which is here defined as the number of units of work FL contained in one unit of energy E. Hence, the units of k_E may be written:

$$U(k_E) = \frac{FL}{E} \tag{4-4}$$

Now, for any mechanical system:

$$E = FL \tag{4-5}$$

whereupon k_E obviously reduces to a dimensionless quantity of unit magnitude.

On the other hand, for any thermal system:

$$E = M\theta \tag{4-6}$$

In this case, k_E reduces to Joule's constant, also known as the mechanical

equivalent of heat and usually denoted by symbol J. Its units become:

$$U(k_E) = U(J) = \frac{FL}{E} = \frac{FL}{M\theta} \qquad (4\text{-}7)$$

Moreover, if the system is not only thermal but also gravitational, which for thermodynamic units is always the case by definition, then we may, if we choose, reduce the units of J further to

$$U(J) = \frac{GL}{G\theta} = \frac{L}{\theta} \qquad (4\text{-}8)$$

which is a dimensional relation that, while correct, is not widely recognized.

The numerical values of Joule's constant for the English and metric units considered in this report are:

English	J = 778.3	ft lbf/Btu	
Metric	J = 4186	joule/kcal	(4-9)

As we have seen, it would also be entirely correct, although unorthodox, to write for the thermal units above the equivalent expressions:

$$1 \text{ Btu} = 1 \text{ lbm } ^{\circ}\text{F}$$

$$1 \text{ kcal} = 1 \text{ kgm } ^{\circ}\text{C} \qquad (4\text{-}10)$$

5. Consistent Derived Units for Dynamic and Thermodynamic Systems

The various auxiliary constraints which characterize dynamic and thermodynamic systems of units were previously summarized in words in Table 2.1. In view of the foregoing development, these same constraints can now be expressed symbolically in the form of specific dimensional relations. This is summarized in Table 5.1.

Table 5.1 Auxiliary Constraints

I. Dynamic Systems

1. Inertial Constraint

$$F = \frac{ML}{T^2}$$

$$M = \frac{FT^2}{L}$$

Either one applies

2. Mechanical Constraint

$$F = FL$$

II. Thermodynamic Systems

1. Gravitational Constraint

$$F = M = G$$

2. Thermal Constraint

$$E = M\theta$$ (A correct but unorthodox relation.)

First, consider the dynamic system of units. By introducing the relations in the upper part of Table 5.1 into the various generalized units originally listed in Table 1.2 we obtain the results summarized in Table 5.2. Note that there are two possibilities depending on whether we choose to eliminate F or M from the list of fundamental (independent) units. These two representations, although distinct, are of course completely equivalent. In other words, we have here two alternative forms of description of one common set of units. These two forms are summarized in Table 5.2.

Next, consider the thermodynamic system of units. By introducing the relations in the lower part of Table 5.1 into the various generalized units listed in Table 1.2 we obtain the results summarized in Table 5.3. Two forms are listed corresponding to the two possible conventions regarding force and mass units.

A comparison of the generalized units in Tables 5.2 and 5.3 show that while some of the quantities have identical generalized dimensions in both the dynamic and thermodynamic systems, others have quite different units in the two systems. Such differences are entirely proper and do not indicate anything amiss.

Nevertheless, for purposes of the subsequent dimensional analysis, it is rather inconvenient to have to deal separately with two distinct and somewhat disparate systems. It would be much more convenient if these two systems could be placed on some common footing so that a single unified analysis could subsequently be made which would be equally applicable to both.

Fortunately, it happens that this can indeed be accomplished, and in a very simple way at that. Two different cases must be considered corresponding respectively to English and metric units. Note from Table 2.2

Table 5.2 Typical Derived Units in Any Dynamic System

	Quantity	Symbol	Generalized Units M, L, T, θ	Generalized Units F, L, T, θ
1.	Angle	α	1	1
2.	Strain	ϵ	1	1
3.	Area	A	L^2	L^2
4.	Volume	v	L^3	L^3
5.	Velocity	V	L/T	L/T
6.	AngularVelocity	ω	1/T	1/T
7.	Acceleration	a	L/T^2	L/T^2
8.	Volumetric Flow Rate	Q	L^3/T	L^3/T
9.	Pressure	p	M/LT^2	F/L^2
10.	Moment	M	ML^2/T^2	FL
11.	Surface Tension	σ	M/T^2	F/L
12.	Viscosity	μ	M/LT	FT/L^2
13.	Kinematic Viscosity	υ	L^2/T	L^2/T
14.	Specific Impulse	I	L/T	L/T
15.	Mass Flow Rate	$\dot{m}$	M/T	FT/L
16.	Mass Flux	G	M/TL^2	FT/L^3
17.	Moment of Inertia	J	ML^2	FLT^2
18.	Density	ρ	M/L^3	FT^2/L^4
19.	Gas Constant	R	$L^2/T^2\theta$	$L^2/T^2\theta$
20.	Specific Weight	γ	M/L^2T^2	F/L^3
21.	Power	P	ML^2/T^3	FL/T
22.	Energy Flux	q/A	M/T^3	F/TL

23.	Specific Enthalpy	h	L^2/T^2	L^2/T^2
24.	Specific Heat	C_p	$L^2/T^2\theta$	$L^2/T^2\theta$
25.	Specific Entropy	s	$L^2/T^2\theta$	$L^2/T^2\theta$
26.	Thermal Conductivity	k	$ML/T^3\theta$	$F/T\theta$
27.	Heat Transfer Coefficient	U	$M/T^3\theta$	$F/TL\theta$
28.	Coefficient of Thermal Expansion	β	$1/\theta$	$1/\theta$
29.	Inertial Constant	k_I	1	1
30.	Work/Energy Factor	k_E	1	1

Table 5.3 Typical Derived Units in Any Thermodynamic System

	Quantity	Symbol	Generalized Units F, M, L, T, θ	Generalized Units G, L, T, θ
1.	Angle	α	1	1
2.	Strain	ϵ	1	1
3.	Area	A	L^2	L^2
4.	Volume	v	L^3	L^3
5.	Velocity	V	L/T	L/T
6.	Angular Velocity	ω	$1/T$	$1/T$
7.	Acceleration	a	L/T^2	L/T^2
8.	Volumetric Flow Rate	Q	L^3/T	L^3/T
9.	Pressure	p	F/L^2	G/L^2
10.	Moment	M	FL	GL
11.	Surface Tension	σ	F/L	G/L
12.	Viscosity	μ	FT/L^2	GT/L^2
13.	Kinematic Viscosity	υ	$(\frac{F}{M})LT$	LT
14.	Specific Impulse	I	$(\frac{F}{M})T$	T
15.	Mass Flow Rate	$\dot{m}$	M/T	G/T
16.	Mass Flux	G	M/TL^2	G/TL^2
17.	Moment of Inertia	J	ML^2	GL^2
18.	Density	ρ	M/L^3	G/L^3
19.	Gas Constant	R	$(\frac{F}{M})L/\theta$	L/θ
20.	Specific Weight	γ	F/L^3	G/L^3
21.	Power	P	$M\theta/T$	$G\theta/T$
22.	Energy Flux	q/A	$M\theta/TL^2$	$G\theta/TL^2$
23.	Specific Enthalpy	h	θ	θ

24.	Specific Heat	C_p	1	1
25.	Specific Entropy	s	1	1
26.	Thermal Conductivity	k	M/TL	G/TL
27.	Heat Transfer Coefficient	U	M/TL^2	G/TL^2
28.	Coefficient Thermal Expansion	β	$1/\theta$	$1/\theta$
29.	Inertial Constant	$k_I = g_o$	$(\frac{M}{F})L/T^2$	L/T^2
30.	Work/Energy Factor	$k_E = J$	$(\frac{F}{M})L/\theta$	L/θ

that for English units, the dynamic and thermodynamic systems share the common units F, L, T, and θ but differ in the units used for M and E. Let the gravitational unit of mass be M' and the gravitational unit of energy be E'. Also note that the units of the conversion factors g_o and J may be written:

$$U(g_o) = \frac{M'}{F}\,\frac{L}{T^2} = \frac{M'}{M} \qquad (5\text{-}1)$$

$$U(J) = \frac{FL}{E'}$$

Now, let X denot any arbitrary quantity as expressed in the dynamic system of units and let X' denote the corresponding quantity as expressed in the related English thermodynamic system. As we have seen, the generalized units of X and of X' may in general be different. Nevertheless, a simple relation can always be found of the form

$$X = g_o^{\,m}\, J^n\, X' \qquad (5\text{-}2)$$

which may be said to convert X' to X. The exponents m and n can always be found such that the units on both sides of Equation (5-2) agree exactly. Note that g_o and J are the only conversion factors required for this purpose.

The procedure involved can best be explained by means of a specific example. Let us take the case of thermal conductivity k, for instance. Then, Equation (5-2) becomes:

$$k = g_o^{\,m}\, J^n\, k' \qquad (5\text{-}3)$$

We next write the corresponding relation of units. It is actually most convenient for this purpose to utilize the original unconstrained forms listed in Table 1.2. Thus:

$$\left[\frac{E}{TL\theta}\right] = \left[\frac{M'}{M}\right]^m \left[\frac{FL}{E'}\right]^n \left[\frac{E'}{TL\theta}\right] \qquad (5\text{-}4)$$

Matching exponents of M' and E', respectively, gives:

$$M': \quad m + 0 + 0 = 0$$

$$E': \quad 0 - n + 1 = 0 \qquad (5\text{-}5)$$

from which we find that in this case:

$$m = 0 \qquad n = 1 \qquad (5\text{-}6)$$

Consequently, Equation (5-3) can finally be written as:

$$k = Jk' \qquad (5\text{-}7)$$

A similar procedure can be applied to all other quantities listed in Table 1.2. This has been done and the results are summarized in the appropriately designated column of Table 5.4.

Next, consider metric units. Table 2.2 shows that in this case the dynamic and thermodynamic systems share in common the units M, L, T, and θ but differ in the units of F and E. Let the distinct gravitational units now be denoted by F" and E". Hence, the units of the conversion factors g_o and J may be written:

$$U(g_o) = \frac{M}{F''}\frac{L}{T^2} = \frac{F}{F''} \qquad (5\text{-}8)$$

$$U(J) = \frac{F''L}{E''}$$

Let X again be an arbitrary quantity as expressed in the dynamic system of units and let X" be the corresponding quantity as expressed in the related metric thermodynamic system. Once more there exists a conversion of the form:

$$X = g_o^{\,m} J^n X'' \qquad (5\text{-}9)$$

Let us illustrate this by an example. Again we choose thermal conductivity k for this purpose. Hence:

$$k = g_o^{\,m} J^n k'' \qquad (5\text{-}10)$$

The corresponding relation of generalized units becomes:

$$\left[\frac{E}{TL\theta}\right] = \left[\frac{F}{F''}\right]^m \left[\frac{F''L}{E''}\right]^n \left[\frac{E''}{TL\theta}\right] \qquad (5\text{-}11)$$

Matching exponents of like terms gives

$$F'': \quad -m + n + 0 = 0 \qquad (5\text{-}12)$$

$$E'': \quad 0 - n + 1 = 0$$

which gives

$$m = n = 1 \qquad (5\text{-}13)$$

Therefore, Equation (5-10) becomes finally:

$$k = g_o \, J \, k'' \qquad (5\text{-}14)$$

The various conversion factors obtained by this method are summarized in Table 5.4. Note that the conversions required in English and in metric units may or may not be of the same form, depending on the quantity in question, and that some of the quantities already have the required units and do not need conversion.

As a result of these conversions, we can confine our attention in the remainder of this report exclusively to the dynamic system of units. Every result obtained for this system now has an exact counterpart in the thermodynamic system. This can always be accomplished simply by substituting for all dynamic quantities their respective thermodynamic equivalents as listed in Table 5.4.

Table 5.4 Conversion Factors

	Quantity	Symbol Dynamic Units	Dimensional Equivalent in Thermodynamic Units: English	Metric
1.	Angle	α	α'	α''
2.	Strain	ϵ	ϵ'	ϵ''
3.	Area	A	A'	A''
4.	Volume	v	v'	v''
5.	Velocity	V	V'	V''
6.	Angular Velocity	ω	ω'	ω''
7.	Acceleration	a	a'	a''
8.	Volumetric Flow Rate	Q	Q'	Q''
9.	Pressure	p	p'	$g_o p''$
10.	Moment	M	M'	$g_o M''$
11.	Surface Tension	σ	σ'	$g_o \sigma''$
12.	Viscosity	μ	μ'	$g_o \mu''$
13.	Kinematic Viscosity	υ	υ'	υ''
14.	Specific Impulse	I	$g_o I'$	$g_o I''$
15.	Mass Flow Rate	$\dot{m}$	$\dot{m}'/g_o$	$\dot{m}''$
16.	Mass Flux	G	G'/g_o	G''
17.	Moment of Inertia	J	J'/g_o	J''
18.	Density	ρ	ρ'/g_o	ρ''
19.	Gas Constant	R	$g_o R'$	$g_o R''$
20.	Specific Weight	γ	γ'	$g_o \gamma''$
21.	Power	P	JP'	$g_o JP''$
22.	Energy Flux	(q/A)	$J(q/A)'$	$g_o J(q/A)''$
23.	Specific Enthalpy	h	$g_o Jh'$	$g_o Jh''$

24.	Specific Heat	C_p	g_oJC_p'	g_oJC_p''
25.	Specific Entropy	s	g_oJs'	g_oJs''
26.	Thermal Conductivity	k	Jk'	g_oJk''
27.	Heat Transfer Coefficient	U	JU'	g_oJU''
28.	Coefficient of Thermal Expansion	β	β'	β''
29.	Inertial Constant	$k_I = g_o$	See note below	
30.	Work/Energy Factor	$k_E = J$		

NOTE: The inertial constant g_o and the work/energy factor (Joule's constant) J provide the needed conversions but, of course, are not themselves converted.

6. Natural Units

Recall that any dynamic system of units has four degrees of freedom. In other words, we may specify the magnitudes of its four fundamental units arbitrarily. These fundamental units are customarily taken either as F, L, T, and θ or as M, L, T, and θ. For definiteness in the ensuing discussion we arbitrarily choose F, L, T, and θ as fundamental. We shall define these units in a special way and call them natural units. We shall designate these fundamental natural units by the special symbols F^*, L^*, T^*, and θ^*.

Natural units are any set of consistent units which are defined on the basis of certain selected dimensional parameters which occur in a problem or phenomena of interest and which are of fundamental significance to the physical situation. To establish a complete basis for a dynamic system of natural units with four degrees of freedom, we require four suitable reference parameters.

The method can be most simply and clearly explained by means of a specific example. Let us take for our example a flow situation which is characterized by four important parameters listed in Table 6.1.

Table 6.1 Reference Parameters

Characteristic Parameter	Symbol	Units
A density	ρ	$(\frac{FT^2}{L^4})$
A velocity	V	$(\frac{L}{T})$
A length	ℓ	L
A thermal conductivity	k	$(\frac{F}{T\theta})$

All of the quantities listed in Table 6.1 are initially expressed not in natural units but in ordinary English or metric units. The symbols F, L, T, and θ without asterisks denote these fixed units.

Now the natural unit of force F* is defined by a relation of the form:

$$F^* = \left[\rho\left(\frac{FT^2}{L^4}\right)\right]^a \left[V\left(\frac{L}{T}\right)\right]^b \left[\ell(L)\right]^c \left[k\left(\frac{F}{T\theta}\right)\right]^d \qquad (6\text{-}1)$$

$$= f_1 F$$

where the four exponents a, b, c, and d remain to be determined. They are determined from the condition that the net exponent of F must equal unity in this case, while the net exponents of L, T, and θ must each be zero. Thus, for each dimension in turn we obtain:

$$\begin{aligned} F&: \quad a + 0 + 0 + d = 1 \\ L&: -4a + b + c + 0 = 0 \\ T&: +2a - n + 0 - d = 0 \\ \theta&: \quad 0 + 0 + 0 - d = 0 \end{aligned} \qquad (6\text{-}2)$$

Solving Equations (6-2) gives:

$$a = 1 \quad b = 2 \quad c = 2 \quad d = 0$$

and
$$F^* = f_1 F = (\rho V^2 \ell^2)F \qquad (6\text{-}3)$$

Similarly, the natural unit of length L* is defined by a relation of the form:

$$L^* = \left[\rho\left(\frac{FT^2}{L^4}\right)\right]^a \left[V\left(\frac{L}{T}\right)\right]^b \left[\ell(L)\right]^c \left[k\left(\frac{F}{T\theta}\right)\right]^d \qquad (6\text{-}4)$$

$$= \ell_1 L$$

Equating coefficients of like terms in similar fashion gives:

$$\begin{aligned} F&: \quad a + 0 + 0 + d = 0 \\ L&: -4a + b + c + 0 = +1 \\ T&: +2a - b + 0 - d = 0 \\ \theta&: \quad 0 + 0 + 0 - d = 0 \end{aligned} \qquad (6\text{-}5)$$

Notice that the left side of Equation (6-5) is indentical to that of Equation (6-2). Only the right side changes to reflect the change in the desired dimension. This will continue to be the case as long as we retain the same set of four reference parameters ρ, V, ℓ, and k.

Solving Equation (6-5) gives:

$$a = 0 \qquad b = 0 \qquad c = +1 \qquad d = 0$$

and (6-6)

$$L^* = \ell_1 L = \ell L$$

The same method may be applied twice more to obtain T* and θ*. Details are left as an exercise for the student. The results are summarized in Table 6.2.

Table 6.2 Natural Units

$$F^* = f_1 F = (\rho V^2 \ell^2) F$$

$$L^* = \ell_1 L = (\ell) L$$

$$T^* = t_1 T = (\frac{\ell}{V}) T$$

$$\theta^* = \tau_1 \theta = (\frac{\rho V^3 \ell}{k}) \theta$$

It is evident that since we now know the four fundamental natural units F*, L*, T*, θ* it becomes a simple matter to express any consistent derived unit in this system. For example, let us derive the natural unit of viscosity. We may write this directly from its basic definition in the form:

$$U^*(\mu) = \frac{F^* T^*}{L^{*2}} = \frac{[(\rho V^2 \ell^2) F][(\frac{\ell}{V}) T]}{[(\ell) L]^2}$$

$$= \rho V \ell \; (\frac{FT}{L^2}) \qquad (6\text{-}7)$$

Alternatively, we may express it in the form:

$$U^*(\mu) = \left[\rho\frac{FT^2}{L}4)\right]^a \left[V(\frac{L}{T})\right]^b \left[\ell(L)\right]^c \left[k(\frac{F}{T\theta})\right]^d \qquad (6\text{-}8)$$

$$= \mu_1(\frac{FT}{L}2)$$

Again equating exponents of like terms as before gives:

$$\begin{aligned} F&: \quad a + 0 + 0 + d = 1 \\ L&: \; -4a + b + c + 0 = -2 \\ T&: \; +2a - b + 0 - d = +1 \\ \theta&: \quad 0 + 0 + 0 - d = 0 \end{aligned} \qquad (6\text{-}9)$$

Solving Equation (6-6) gives:

$$a = +1 \quad b = +1 \quad c = +1 \quad d = 0$$

and (6-10)

$$U^*(\mu) = \mu_1(\frac{FT}{L}2) = \rho V\ell\ (\frac{FT}{L}2)$$

Comparison of Equation (6-7) and Equation (6-10) shows that the results obtained by these two methods are identical.

Notice that Equations (6-2), (6-5), and (6-9) can be solved because the determinant of the coefficients on the left is nonvanishing, that is:

$$\begin{vmatrix} +1 & 0 & 0 & +1 \\ -4 & +1 & +1 & 0 \\ +2 & -1 & +0 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} \neq 0 \qquad (6\text{-}11)$$

If this inequality (6-11) had not been satisfied, this would have signified that the four equations involved were not all linearly independent, so that no solution could have been obtained. It would have meant that the original choice of reference parameters did not constitute an adequate set. In that case it would have been necessary to change one or more of the reference parameters. As it is, the parameters ρ, V, ℓ, k actually chosen do constitute an adequate set

Now having a complete and definite set of consistent natural units at our disposal, we can express any dimensional quantity of interest in terms of these units. This idea can best be explained by means of one or two specific examples. For the first example consider some force f which happens to be of interest. Expressing this same force first in fixed units then in natural units and equating the two gives:

$$fF = f^*F^* \tag{6-11}$$

Now writing F* in terms of the relevant reference parameters gives:

$$fF = f^*F^* = f^*(\rho V^2 \ell^2)F \tag{6-12}$$

Hence:

$$f^* = \frac{fF}{F^*} = \frac{fF}{(\rho V^2 \ell^2)F} = \frac{f}{\rho V^2 \ell^2} \tag{6-13}$$

Notice that the fixed unit F cancells out of Equation (6-13) so that f*, the force as expressed in natural units, turns out to be dimensionless.

Take another example. Let μ denote the viscosity of the fluid. Expressed in natural units this becomes

$$\mu^* = \frac{\mu\left(\frac{FT}{L^2}\right)}{\left(\frac{F^*T^*}{L^{*2}}\right)} = \frac{\mu\left(\frac{FT}{L^2}\right)}{\rho V \ell \left(\frac{FT}{L^2}\right)}$$

or (6-14)

$$\mu^* = \frac{\mu}{\rho V \ell}$$

The reader will recognize that the viscosity μ^* as expressed in natural units is simply the reciprocal of the familiar Reynolds number. This provides a useful and interesting interpretation of the physical significance of Reynolds number. In other words, Reynold's number is nothing more than an indication in generalized terms of the relative importance and influence of viscosity in a given type of flow field, high Reynold's number denoting low viscosity and vice versa.

Notice that μ^*, like f^*, is dimensionless. It is apparent from the manner of their derivation that all initially dimensional quantities, when finally expressed in natural units, are thereby reduced to dimensionless form. Moreover, being dimensionless, the magnitudes of these quantities become independent of any particular fixed units! For example, the same value is obtained for μ^* whether μ, ρ, V, and ℓ are expressed in English units or in metric units. This means that when expressed in consistent natural units, all quantities are reduced to their most fundamental and invariant form.

It is also of interest to express the four reference quantities themselves in dimensionless form. Working first with density ρ, we obtain the following relation where the necessary exponents are supplied immediately by simple inspection.

$$\rho^* = \frac{\left[\rho\left(\frac{FT^2}{L^4}\right)\right]}{\left[\rho\left(\frac{FT^2}{L^4}\right)\right]^1 \left[V\left(\frac{L}{T}\right)\right]^0 \left[\ell(L)\right]^0 \left[k\left(\frac{F}{TL}\right)\right]^0}$$

$$= 1 \qquad (6\text{-}15)$$

Analogous results are obtained for the other reference parameters so that we can write:

$$\rho^* = V^* = \ell^* = k^* = 1 \qquad (6\text{-}16)$$

Suppose that the four reference parameters, instead of being ρ, V, ℓ, and k, happen to be four other parameters, call them A, B, C, and D. Also let X be any quantity of arbitrary dimensions. Then X can always be transformed into a dimensionless version X* in natural units according to a relation of the general form:

$$X^* = \frac{X}{A^a B^b C^c D^d} \qquad (6\text{-}17)$$

By using the methods previously explained, a definite solution can always

be obtained for the four exponents a, b, c, d such that the numerator and denominator of (6-17) are both of like dimension. Hence, X^* is dimensionless. The denominator of (6-17) then expresses the natural unit of X as a numerical multiple of the corresponding ordinary fixed unit.

It is not difficult to see from Equations (6-16) and 6-17) that however the reference parameters A, B, C, D be chosen, when they themselves are expressed in the natural system of units, they will invariably be reduced to unit magnitudes, that is:

$$A^* = B^* = C^* = D^* = 1 \qquad (6\text{-}18)$$

In fact, the natural system of units may be defined as that system of consistent dynamic units in which the four reference parameters themselves take on unit magnitudes.

Naturally, the four reference parameters A, B, C, D must always be so chosen that the determinant analogous to (6-11) is nonvanishing. Fortunately, this is only a very mild constraint.

7. Formal Invariance of Physical Equations

Consider any physical equation which is valid when all quantities in it are expressed in fixed English or metric units of the type we have defined as comprising a dynamic system. Now let us ask what happens if we shift from these initial units to some other set, such as from English to metric, or vice versa. We stipulate, however, that both of these sets of units shall conform to the constraints which define a dynamic system. In other words, the shift amounts simply to arbitrary changes in the magnitudes of the four fundamental units, nothing more. Naturally, the magnitudes of all consistent derived units then also change correspondingly, in accordance with the fixed relations of consistency implied by their respective generalized labels.

What is the effect on our mathematical equation of such arbitrary but systematic changes in the actual magnitudes of all units? In this connection we note firstly that all additive terms of any valid equation are always expressed in identical units. Thus, while a shift in the size of the units will affect the numerical magnitudes of every additive term in any given equation, the numerical values of all such terms will always be changed by exactly the _same_ numerical conversion factor! Hence, an equation that is initially satisfied in terms of the original units will continue to be satisfied when expressed in terms of the new units. Moreover, the mathematical form of the equation itself, when expressed in terms of symbols rather than numbers, remains entirely unaffected by this type of change. This fact we term "the principle of formal invariance of physical equations."

These ideas can perhaps be made most clear by means of a simple example. Consider the formula for the volume V of a sphere of radius r. This may be written:

$$V = \frac{4}{3} \pi r^3 \qquad (7\text{-}1)$$

The consistent generalized units in this case are unit length L for r and unit volume L^3 for V. Now the mathematical formula (7-1) itself remains valid and unchanged in analytical form regardless of how the unit length L happens to be specified, whether in inches, feet, centimeters, etc. For a sphere of fixed physical size, the numerical magnitudes of the terms on both sides of the equation do indeed depend on the units used, but the analytical form of the equation itself is in no way affected by the choice of units.

The reader can verify that these same relations so clearly illustrated by this simple geometrical example are in fact true in general.

Since natural units also satisfy all of the requirements stipulated above, we conclude that if all quantities in any valid physical equation be transformed from their initial fixed units to corresponding natural units, the equation itself continues to be valid and its mathematical form is not affected by this change of dimensional base.

8. The Pi Theorem

With these principles established, it now becomes a simple matter to verify the validity and clarify the significance of a famous theorem of dimensional analysis known as the Pi Theorem.

Consider any set of n physical quantities which may be of significance in a particular physical situation. Let a subset of four of these quantities, denoted by A, B, C, D, represent the parameters which will be used to establish the dynamic system of natural units. Let the remaining quantities be denoted simply as X_1, X_2, -- X_i---X_{n-4}. All of the foregoing are initially expressed in ordinary fixed English or metric units. Let there exist among these parameters one or more known or unknown physical equations or physical relationships which we symbolize here in the form:

$$f(A, B, C, D; X_1, X_2, \text{---} X_i, \text{----} X_{n-4}) = 0 \qquad (8\text{-}1)$$

As we have already seen, the physical parameters X_1, X_2---X_i----X_{n-4} can be reduced to natural units by relations of the form:

$$X_i^* = \frac{X_i}{A^{a_i} B^{b_i} C^{c_i} D^{d_i}} \qquad (8\text{-}2)$$

$$i = 1, 2, \text{----}(n-4)$$

where the exponents a_i, b_i, c_i, d_i can always be found such as to render the X_i^* dimensionless. Also, as we have seen, the reference parameters themselves will transform to unit magnitudes, that is:

$$A^* = B^* = C^* = D^* = 1 \qquad (8\text{-}3)$$

Moreover, according to the principle of formal invariance of physical equations, the symbolic mathematical form of relation (8-1), whether known or unknown, will not be affected in any way by the shift to natural units. Hence, Equation (8-1) will now assume the form:

$$f(1, 1, 1, 1; X_1^*, X_2^*, \text{---} X_i^*, \text{---} X_{n-4}^*) = 0 \qquad (8\text{-}4)$$

This result shows how a situation which is initially defined by n dimensional parameters can always be reduced to an equivalent description in terms of (n-4) dimensionless parameters. This is the essence of the Pi Theorem.

In the discussion so far, we have considered only the case of a system with just four degrees of freedom. Other cases can arise involving some greater or lesser number of degrees of freedom. For example, if we extend the present theory to include electrical charge as an additional fundamental dimension, the number of degrees of freedom can increase to five. Also, if we choose to deal with dimensional systems involving either more or fewer auxiliary constraints than those which characterize the present system of dynamic units, the number of degrees of freedom can either decrease or increase. Also within the specific theoretical framework here developed, we can have problems which do not require all four of the presently available degrees of freedom. For example, there are many problems in mechanics in which phenomena involving temperature happen to be of no interest. In that case, we may drop θ from the list of fundamental dimensions required and consequently reduce the number of reference parameters needed from four to three.

The previous reasoning can be readily generalized to fit such varying circumstances. Suppose, for example, that we are dealing with a situation that involves n dimensional parameters in all and requires k fundamental dimensions. From the pattern already established, it is apparent that a subset consisting of k of the dimensional parameters can be chosen as the basis for the system of natural units. These k parameters take on unit magnitudes when themselves transformed into natural units. The remaining (n-k) parameters become transformed into dimensionless form. It is common in dimensional analysis to refer to such dimensionless parameters as

dimensionless pi's. Hence, the situation may be summarized by saying that a set of n dimensional parameters involving k fundamental dimensions can always be reduced to an equivalent set of (n-k) dimensionless pi's. This statement is known as the Pi Theorem.

The great practical importance of the Pi Theorem stems from two inter-related facts. Firstly, this theorem permits a reduction in the number of significant parameters which must be considered in any physical problem from n to (n-k). This is usually a tremendous simplification in itself. In experimental work, for instance, it can represent an enormous savings of effort, time, and money. Secondly, the final parameters which are ultimately retained turn out to be dimensionless. This means that they are expressed in their most general, significant, and invariant form, all nonessential aspects having been eliminated through the nondimensionalizing process. This is also of the greatest value in enhancing theoretical insight.

In some textbook discussions of the Pi Theorem, the somewhat misleading impression is created that this method is applicable mainly to experimental work, as a kind of gimmick which is only useful for presenting test curves. On the contrary, the approach via natural units is of the greatest value also for theoretical work. In particular, the systematic reduction of all analytical equations to their appropriate dimensionless forms in accordance with the principles of natural units and of the formal invariance of equations adds immeasurably to the simplicity, power, significance, and elegance of the analysis. Organization of all results on this basis is mandatory for all serious scientific work, both experimental and theoretical.

9. On Choosing Reference Parameters

In solving any given problem, the dimensional reference parameters A, B, C, D should be selected so far as possible from among the parameters that are:

1) highly significant.

2) independent or known.

3) relatively constant.

4) finite and nonvanishing.

Once a definite set of reference parameters has been selected, it is usually advisable to adhere to this set consistently throughout the entire course of a given problem. This guarantees that all results will be expressed on the basis of a single consistent and known set of natural units. A consistent approach of this kind represents a very powerful method of imposing the maximum possible degree of simplicity and order in the analysis of any complex physical phenomenon.

It is likewise desireable that every equation be expressed consistently in terms of the corresponding natural units, with each quantity expressed in its dimensionless version. When this is done, each relation is displayed in its most general and significant form, and all unessential aspects are eliminated. Moreover, all final numerical results then become wholly independent of the particular system of fixed units, whether English or metric, in which all dimensional quantities happen to be expressed initially.

10. Some Typical Applications

Our first example closely parallels the case treated in Section 6. It deals with the flow of fluids at essentially constant density. This restriction on density has the effect of eliminating any direct influence of heat or temperature on the mechanics of the flow. Hence, in an inertial system, if we are not interested in heat transfer effects, only three fundamental units are involved in the mechanics of the problem. For definiteness, we take these to be F, L, and T. Hence, three fundamental reference parameters are required for constructing the system of natural units. The pertinent physical properties are density ρ and viscosity μ. However, under normal circumstances inertial effects are of far greater magnitude than viscous effects. Therefore, in this case, density ρ must be regarded as playing the more fundamental role with viscosity μ representing merely a modifying influence.

For any given type of geometrical configuration, for example, flow about an aircraft model of given design, it is necessary to choose some characteristic length ℓ to represent the scale of size involved. Thus, if the aerodynamics of the wing are considered to be of dominant importance, some characteristic dimension of the wing such as wing span b or mean geometric chord c might be chosen for this purpose.

As a rule, in most fluid mechanics problems, there also exists some velocity V which characterizes the kinematics of the field in a natural way. Thus, in flow about an aircraft model, the velocity of the undisturbed fluid far from the model may be chosen for this purpose. For flow through a uniform pipe, the volumetric mean velocity represents a suitable choice.

It is clear, therefore, that density ρ, characteristic length ℓ, and characteristic velocity V constitute the natural reference parameters

for a vast range of fluid mechanics problems of the general type just described.

In problems of this type, we are often interested in evaluating certain overall forces such as the lift or drag on the airplane model. In other cases we might wish to evaluate certain pressures or stresses, such as the shear stress at the wall of a pipe. In most cases, the above forces and stresses will be influenced to some degree by the visosity of the fluid. Hence, our problem relates to quantities like those illustrated in Table 10.1.

We illustrate the procedure for establishing natural units for this case by considering in detail the fundamental unit of force F*. This may be represented in the form:

$$F^* = \left[\rho\left(\frac{FT^2}{L^4}\right)\right]^a \left[V\left(\frac{L}{T}\right)\right]^b \left[\ell(L)\right]^c \qquad (10\text{-}1)$$

We now equate exponents of like units. Thus:

$$\begin{array}{ll} \text{for } F & a + 0 + 0 = +1 \\ \text{for } L & -4a + b + c = 0 \\ \text{for } T & +2a - b + 0 = 0 \end{array} \qquad (10\text{-}2)$$

The solution is:

$$a = 1 \qquad b = 2 \qquad c = 2 \qquad (10\text{-}3)$$

whereupon the required unit of force becomes:

$$F^* = \rho\, V^2\, \ell^2\, F \qquad (10\text{-}4)$$

The other results in Table 10.1 are obtained by the same general method.

Two features of Table 10.1 warrant reiteration. Note again that when transformed into dimensionless pi's, the reference parameters transform into unit magnitudes. Also, notice again that the dimensionless pi corresponding to viscosity turns out to be the reciprocal of the Reynolds

Table 10.1 Typical Physical Quantities in Incompressible Flow

Quantity	Symbol	Fixed Unit	Ratio of Natural Unit to Fixed Unit	Dimensionless Pi
Reference Parameters				
Density	ρ	FT^2/L^4	ρ	1
Velocity	V	L/T	V	1
Length	ℓ	L	ℓ	1
Other Quantities				
Force	f	F	$\rho V^2 \ell^2$	$f^* = \frac{f}{\rho V^2 \ell^2}$
Stress	τ	F/L^2	ρV^2	$\tau^* = \frac{\tau}{\rho V^2}$
Viscosity	μ	FT/L^2	$\rho V \ell$	$\mu^* = \frac{\mu}{\rho V \ell}$

Fundamental Natural Units

Force	$F^* = (\rho V^2 \ell^2)F$
Length	$L^* = (\ell)L$
Time	$T^* = (\frac{\ell}{V})T$

number. Conversely, Reynolds number is merely the reciprocal of the viscosity as expressed in the ρ, V, ℓ system of natural units.

The usefulness of the natural units can be shown in yet another way. Suppose we are investigating experimentally the drag force D on a certain aircraft configuration. The drag D will depend not only on the shape and attitude of the model, but also on the parameters ρ, V, ℓ, and μ. We may express this fact symbolically in the form:

$$D = f(\rho, V, \ell, \mu) \qquad (10\text{-}5)$$

Upon invoking the principle of the formal invariance of physical euations, we may immediately rewrite this in terms of natural units in the form:

$$D^* = f(1, 1, 1, \mu^*) \qquad (10\text{-}6)$$

Since ρ, V, and ℓ all transform to unity, they become constants and no longer need be referred to explicitly. Hence, Equation (10-6) is equivalent to:

$$\left(\frac{D}{\rho V^2 \ell^2}\right) = f'\left(\frac{\mu}{\rho V \ell}\right) \qquad (10\text{-}7)$$

By departing slightly from strict natural units, we may put this in the form more usually encountered, namely:

$$\left(\frac{D}{\frac{1}{2}\rho V^2 \ell^2}\right) = f''\left(\frac{\rho V \ell}{\mu}\right) \qquad (10\text{-}8)$$

The factor $\frac{1}{2}$ is inserted on the left in recognition of the fact that the quantity $\frac{1}{2}\rho V^2$ occurs in Bernoulli's equation and represents the so-called dynamic pressure. The dimensionless drag force on the left of Equation (10-8) is termed the drag coefficient and is usually denoted by C_D. The dimensionless parameter on the right is simply the Reynolds number, often written as Re. Hence, Equation (10-8) may be rewritten simply as:

$$C_D = f''(\text{Re}) \qquad (10\text{-}9)$$

The transformation involved in going from Equation (10-5) to Equation (10-9) is of course exactly consistent with the Pi Theorem. Notice the great simplification entailed in reducing the number of significant parameters from five to two.

If all quantities in the above problem happen to be initially expressed in English gravitational units, it is first necessary to convert them to English inertial units before proceeding. From Table 5.4 we see, however, that the only parameter in the above list which happens to be affected by this change is the density. The required conversion is simply:

$$\rho = \rho'/g_o \qquad (10\text{-}10)$$

The procedure beyond this point is now exactly as before and need not be repeated here. It is of interest, however, to inspect the result so obtained for μ^*, when its components are expressed in gravitational units, namely:

$$\mu^* = \frac{\mu'}{(\rho'/g_o)V'\ell'} = \frac{1}{Re} \qquad (10\text{-}11)$$

Some writers prefer to rearrange this to read

$$Re = \frac{\rho'V'\ell'}{(g_o\mu')} \qquad (10\text{-}12)$$

thereby associating the conversion factor g_o with μ' rather than with ρ'. Now μ' has the generalized units FT/L^2 whereas $(g_o\mu')$ has the units M/LT. Consequently, viscosity in gravitational units is sometimes expressed in the units M/LT rather than FT/L^2. Note, also, that in any inertial system the units FT/L^2 and M/TL are exactly equivalent; in a gravitational system they are not. Incidentally, the units FT/L^2 correspond exactly to the usual definition of viscosity as the ratio of stress to strain rate; the

gravitational units M/TL do not. Nevertheless, both units are found in practice. There need be no confusion over this provided that the actual units used are clearly labelled.

Under certain special conditions, the inertial forces, instead of being much greater than the viscous forces, become much smaller. This is true, for example, of laminar flow in a uniform pipe. It is also true for any geometrical configuration at a sufficiently low Reynolds number, that is, at a sufficiently high dimensionless viscosity. These are sometimes called creeping flows. In such cases, it becomes advantageous to change the reference parameters from ρ, V, ℓ, to μ, V, ℓ. When this is done, the resulting dimensionless pi's as measured by experiment are found to exhibit a much simpler behavior than if expressed in the ρ, V, ℓ system. Details of this particular case are summarized in Table 10.2. Verification is left as an exercise for the student.

Our next example will deal with turbo pumps for incompressible fluids. The significant fluid property now is clearly density ρ, with viscosity μ playing a subordinate role. We again use inertial units as being simpler. Consider the problem of testing a particular machine from among a family of geometrically similar models which vary only in size. Wheel diameter D can be chosen as the characteristic length. The rotational speed N can be established and controlled independently and is held nearly constant in normal use. Volumetric flow rate Q varies in response to certain valve settings as does the net useful pressure rise through the machine. However, in place of pressure rise, we prefer to utilize the equivalent enthalpy rise per unit mass H. We can now write:

$$H = f(\rho, N, D, Q, \mu) \qquad (10\text{-}13)$$

In this situation ρ, N, D provide the appropriate reference parameters. This choice conforms to the rules given in Section 9.

Table 10.2 Typical Physical Quantities in Creeping Flows

Quantity	Symbol	Fixed Unit	Ratio of Natural Unit to Fixed Unit	Dimensionless Pi
Reference Parameters				
Viscosity	μ	FT/L^2	μ	1
Velocity	V	L/T	V	1
Length	ℓ	L	ℓ	1
Other Quantities				
Force	f	F	$\mu V\ell$	$f^* = F/\mu V\ell$
Stress	τ	F/L^2	$\mu V/\ell$	$\tau^* = \tau\ell/\mu V$
Density	ρ	FT^2/L^4	$\mu/V\ell$	$\rho^* = \rho V\ell/\mu = Re$

Fundamental Natural Units

Force	$F^* = (\mu V\ell)F$
Length	$L^* = (\ell)L$
Time	$T^* = (\frac{\ell}{V})T$

Nondimensionalizing in the usual way gives:

$$\left(\frac{H}{\rho N^2 D^2}\right) = f\left[\left(\frac{Q}{ND}\right), \left(\frac{\mu}{\rho ND^2}\right)\right] \qquad (10\text{-}14)$$

The first two of these dimensionless pi's represent dimensionless enthalpy rise and dimensionless flow rate, respectively, and are clearly of dominating importance. The third pi represents the modifying influence of viscous effects. It is often ignored in practice.

Now consider the problem of turbo-pumps from another viewpoint. Suppose we wish not to test a given machine, but to select a suitable machine to perform a specified pumping job. In this context, the primary knowns would be ρ, H, Q and these become the reference parameters. Equation (10-13) may be rearranged to state that:

$$D = f(\rho, H, Q, N, \mu) \qquad (10\text{-}15)$$

Nondimensionalizing in the usual way gives:

$$\left(\frac{D}{H^{-1/4}Q^{1/2}}\right) = f\left[\left(\frac{N}{H^{3/4}Q^{-1/2}}\right), \left(\frac{\mu}{\rho H^{1/4}Q^{1/2}}\right)\right] \qquad (10\text{-}16)$$

The first two of these pi's are commonly termed the specific diameter and specific speed, respectively. This example shows that fixing the specific speed of a turbo pump largely determines the required specific diameter required. Again viscosity plays a very secondary role.

Suppose now that we deal not with turbo pumps but water turbines. For testing a turbine of given design the appropriate reference parameters would be the density ρ, the wheel diameter D, and the useful enthalpy drop per unit mass H supplied for driving the turbine. In addition, the rotational speed N could be treated as an independent parameter and the resulting shaft power P_s and the mass flow rate $\dot{m}$ as dependent parameters.

On the other hand, if we are selecting a water turbine to perform a specified service, the more convenient reference parameters would be

density ρ, available enthalpy drop per unit mass H, and required shaft power P_s. Then rotational speed N could be the independent parameter and required wheel diameter D and mass flow rate $\dot{m}$ the dependent parameters. It is suggested that the student work out the detailed dimensionless pi's for these two cases.

Next, consider the case of a family of geometrically similar compressors. Suppose that both enthalpy rise per unit mass H and absolute outlet pressure P_2 are functions of absolute inlet pressure P_1, absolute inlet temperature T_1, wheel diameter D, rotational speed N, and mass flow rate $\dot{m}$. Let us neglect viscosity effects in this instance. Notice that temperature is now involved so that four reference parameters are needed. It is suggested that the student, using P_1, T_1, N, and D for this purpose, determine the three dimensionless pi's involved and that he express in symbolic terms the two unknown relations that exist among them.

As another example of the foregoing principles, consider the thrust f produced by an ideal propeller of disc area A when operating in a fluid of density ρ. The propeller is supplied with shaft power P. The relative forward velocity of the propeller with respect to the undisturbed fluid is V.

From the momentum theory of propellers, it is known that the foregoing parameters satisfy the relation.

$$f^3 = 2\rho AP(P - fV) \qquad (10\text{-}17)$$

We are usually interested in the performance of a propeller of known size operating in a known medium and driven by an engine of known power. Hence ρ, A, P are obviously the appropriate reference parameters for this case. Only three reference parameters are needed since temperature θ is not involved in any of the five parameters of Equation (10-17).

The dimensional exponents in the units corresponding to the three reference parameters can now be summarized as follows.

Parameter	Units
ρ	FT^2/L^4
A	L^2
P	FL/T

(10-18)

Let us first determine the fundamental natural unit of force. Thus:

$$F^* = \left[\rho\left(\frac{FT^2}{L^4}\right)\right]^a \left[A(L^2)\right]^b \left[P\left(\frac{FL}{T}\right)\right]^c \qquad (10\text{-}19)$$

Equating exponents of like terms leads to three equations in the exponents of F, L, and T, respectively.

$$F: \quad 1a + (0)b + 1c = 1$$

$$L: \quad -4a + 2b + 1c = 0$$

$$T: \quad +2a + (0)b - 1c = 0 \qquad (10\text{-}20)$$

The solution of these equations gives:

$$a = 1/3$$

$$b = 1/3$$

$$c = 2/3 \qquad (10\text{-}21)$$

The natural unit of force F* is, therefore, related to the fixed unit F as follows.

$$F^* = (\rho^{1/3} A^{1/3} P^{2/3})F \qquad (10\text{-}22)$$

When expressed in this natural unit, the dimensionless thrust, therefore, becomes simply:

$$f^* = \frac{f}{\rho^{1/3} A^{1/3} P^{2/3}} \qquad (10\text{-}23)$$

Upon repeating this procedure for the other parameters, the following results are obtained. The symbol $\rightarrow$ means "is transformed to." Thus:

$$\rho \to \rho^* = 1$$

$$A \to A^* = 1$$

$$P \to P^* = 1$$

$$f \to f^* = \frac{f}{\rho^{1/3}\,A^{1/3}\,P^{2/3}}$$

$$V \to V^* = \frac{V}{\rho^{-1/3}\,A^{-1/3}\,P^{1/3}} \qquad (10\text{-}24)$$

Next we replace the five dimensional quantities in Equation (10-17) by their dimensionless counterparts as defined by Equation (10-24). The results may be summarized in the form:

$$f^{*3} = 2(1 - f^*V^*) \qquad (10\text{-}25)$$

This result expresses the relation between the dimensionless thrust f^* and the dimensionless forward speed V^* for an ideal propeller. It can be shown that any real propeller can approach but never exceed the ideal performance defined by Equation (10-25). Hence, the result in this form is highly general and significant.

Notice that Equation (10-25) is much simpler to grasp and far more informative than the original dimensional version (10-17). However, this transformation can only be made on the basis of the dimensionless parameters defined in Equation (10-24).

Imagine an investigator acquainted with Equation (10-17) but insufficiently versed in the principles of dimensional analysis. It is highly unlikely that he would intuitively hit upon the simple looking yet sophisticated form (10-25), or upon the unfamilar parameters defined in (10-24). Even if these parameters were actually pointed out to him, it is unlikely that he could readily grasp their real significance. Yet any student armed with an understanding of the dimensional principles

explained in this paper can derive these quantities by a straightforward procedure and readily interpret their physical significance.

Of course, when we consider real propellers, the relation (10-25) is modified by various factors including the effects of rotational speed N and of viscosity μ. However, the fact that the quantities N and μ do not even appear in the basic momentum formulation merely confirms that these are indeed secondary rather than primary parameters. If we wish to include them, Equation (10-25) must be replaced by an experimentally determined relation of the form:

$$f^* = f(V^*, N^*, \mu^*) \qquad (10\text{-}26)$$

where the secondary pi's are defined as follows.

$$N^* = \frac{N}{\rho^{-1/3} A^{-5/6} P^{+1/3}} \qquad (10\text{-}27)$$

$$\mu^* = \frac{\mu}{\rho^{2/3} A^{1/6} P^{1/3}}$$

It is of interest to compare the foregoing formulation with a more commonly encountered alternative. For any family of geometrically similar fixed pitch propellers, the thrust and shaft power are determined by two relations of the form:

$$f = f_1(\rho, N, D, V, \mu)$$

$$P = f_2(\rho, N, D, V, \mu) \qquad (10\text{-}28)$$

In the conventional analysis, ρ, N, D are chosen as reference parameters. The results become, in our present notation

$$f^* = f_1(V^*, \mu^*)$$

and

$$P^* = f_2(V^*, \mu^*) \qquad (10\text{-}29)$$

where

$$f^* = \frac{f}{\rho N^2 D^4} = \text{thrust coefficient } (C_F)$$

$$P^* = \frac{P}{\rho N^3 D^5} = \text{power coefficient } (C_P)$$

$$V^* = \frac{V}{ND} = \text{advance ratio } (J)$$

$$\mu^* = \frac{\mu}{\rho N D^2} = \text{viscosity parameter} \qquad (10\text{-}29)$$

The symbol enclosed in parenthesis in each of the above expressions is the conventional symbol for the parameter in question. As a rule, the viscosity parameter μ^* is not included in conventional analyses but is shown here for the sake of completeness.

The above scheme of conventional coefficients is often a very convenient one. However, it does not lend itself to displaying the inherent performance limitation implied by the momentum analysis in the clear and simple form shown in Equation (10-25). Hence, these conventional coefficients are not as fundamental as those defined earlier in Equation (10-24). The basic reason for this limitation is that N, a parameter of relatively secondary importance, is included among the reference parameters.

Our next example is closely related to the previous one. It deals with the power required by an ideal rotorcraft of weight W to climb vertically at a steady rate of climb V. The basic propeller relation Equation (10-17) applies also to this case, except that the rotor thrust f becomes equal to the weight W. However, for this application the quantities ρ, A, and W now constitute the preferred reference parameters. Apart from this, the procedure is the same as before. The following results are obtained.

$\rho \to \rho^* = 1$

$A \to A^* = 1$

$W \to W^* = 1$

$$P \to P^* = \frac{P}{\rho^{-1/2} A^{-1/2} W^{+3/2}} = \text{dimensionless power} \qquad (10\text{-}30)$$

$$V \to V^* = \frac{V}{\rho^{-1/2} A^{-1/2} W^{+1/2}} = \text{dimensionless rate of climb}$$

Now, Equation (10-17) translates to:

$$1 = 2P^* (P^* - V^*) \qquad (10\text{-}31)$$

which fixes the minimum dimensionless power P* required for any specified value of dimensionless rate of climb V*.

Our final example relates to a fixed windmill or small air turbine which extracts useful power from the wind or from the slipstream. The simple momentum energy relation given by Equation (10-17) still applies. However, the sense of the force is reversed as is also the sense of the power flow. To avoid the inconvenience of dealing with negative signs, it is advisable to replace f by -D and P by -P in Equation (10-17). We thereby obtain:

$$D^3 = 2\rho AP (DV-P) \qquad (10\text{-}32)$$

It is now appropriate to choose ρ, A, V as reference parameters. Notice that parameter V was not a suitable reference quantity in the previous applications because it could take on zero values for those cases. However, in the application to the windmill, the wind velocity V must necessarily be nonzero, of course. Hence, the quantity becomes a suitable reference in the present context. We therefore obtain:

$$D^* = \frac{D}{\rho A V^2} \qquad (10\text{-}33)$$

$$P^* = \frac{P}{\rho\ A V^3} \qquad (10\text{-}34)$$

The basic equation (10-32) now reduces to:

$$D^{*3} = 2P^* \, (D^* - P^*) \qquad (10\text{-}35)$$

This fundamental result expresses the limiting dimensionless power P^* attainable from an ideal windmill or air turbine as a function of the dimensionless drag force D^*. This represents a theoretical performance limit which any real device may approach but never exceed. For a small auxiliary power turbine mounted, say on an aircraft, the drag force D^* is of definite interest. For a stationary windmill acted upon by the wind, the drag force would seldom be of much interest in itself; the power available is the only parameter of real concern in this case.

It is suggested that the student sketch the curve of P^* versus D^* from Equation (10-35). It is easy to see that the curve must pass through the origin. By differentiating Equation (10-35) we find that the maximum power point has the coordinates:

$$D^*_{crit} = \left(\frac{2}{3}\right)^2 \qquad P^*_{max} = \left(\frac{2}{3}\right)^3 \qquad (10\text{-}36)$$

while the maximum drag point has the coordinates:

$$D^*_{max} = \frac{1}{2} \qquad P^*_{crit} = \frac{1}{4} \qquad (10\text{-}37)$$

The last few examples above are particularly instructive because they show that an astonishing amount of very clear, valuable, and basic information can be extracted from something as elementary as the basic momentum-energy relation of Equation (10-17). These examples also illustrate the rationale which governs the choice of reference parameters.

The wealth of information and the depth of insight that can be attained by the judicious use of consistent natural units is not as widely nor as fully appreciated as it should be. It is hoped that this discussion has succeeded firstly in explaining clearly the concepts and

procedures involved and secondly in demonstrating the great scope and value of these dimensional methods.

For the convenience of the reader who might wish to pursue this subject further, a bibliography is appended in the next section.

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13. ABSTRACT This monograph presents an introduction to dimensional analysis, a subject of profound significance for all quantitative sciences and engineering. Although the purely formal mathematical relations of dimensional analysis can be found in many texts, the present analysis, while still reasonably brief, goes well beyond the usual textbook treatment. It develops a unified theory of physical measurements which involves several novel and original features. Firstly, the usual qualitative concept of a physical dimension is replaced by a precise quantitative definition. Secondly, the famous Pi Theorem is shown to follow simply from the existance of a certain comprehensive set of consistent "natural units" which differ from ordinary English or metric units only in their fundamental scales of force, length, time, and temperature; these scales can always be chosen to fit the particular phenomena under consideration. Thirdly, it is shown that the mathematical form of any equation remains unaffected if all quantities in it be consistently transformed from ordinary fixed units to natural units. Fourthly, the analysis uncovers an unconventional dimensional relation (energy = mass x temperature) that is valid and useful under certain commonly encountered conditions. Finally, the theoretical discussion clears up certain common conceptual muddles, including those often associated with the various alternative units of force and mass.

Engineering applications of the theory are illustrated by various examples involving mainly fluid flow, turbomachinery, propellers, and related devices. In particular, highly significant absolute performance limits are established for certain broad classes of propellers, lifting rotors, and air turbines, merely by nondimensionalizing the well known momentum/energy relations in an unconventional but systematic manner.

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Generalized Units						
Unit and Measure						
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Thermodynamic Similarity						
Theory of Models						
Reynolds Number						
Momentum Theory of Propellers						
Propeller Parameters						
Turbine Parameters						
Rotorcraft Parameters						
Pump Parameters						
Compressor Parameters						
Performance Limits of:						
Propellers						
Axial Turbines						
Lifting Rotors						
Mathematical Invariance of Physical Equations						